

A Situational-Representational Didactic Design for
Fostering Conceptual Understanding of Mathematical Content:
The Case of Ratio and Proportion¹

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Introduction

The purpose of this paper is to present and discuss the design rationale and results from an empirical study of an innovative instructional intervention in the domain of ratio and proportion conducted in two fifth-grade classrooms ($n=19$, $n=20$). The design was innovative both in its use of the multiplication table as an external representational resource for fostering an understanding of ratio and proportion and in its integration of and reciprocity between real-world situations and formal mathematical structures. The full paper will describe in detail the didactic stages of the curricular unit stemming from the intervention. I will present the design details and methodology of the intervention only to the extent needed to elucidate and articulate the rationale of the designed reciprocity between situations and their symbolic representations. The assertion of this proposal is that students' bilateral constitution of the ratio and proportion concept—as embedded simultaneously in situations and in representations—enabled them to successfully solve missing-value ratio-and-proportion word problems.

The motivation for the intervention was a concern with students' low achievement in the domain of ratio and proportion, and in particular students' robust pattern of “additive” errors that lead to erroneous conjectures such as ‘ $3:5=6:8$ because the additive relations are maintained across the equation’ (Behr, Harel, Post, & Lesh, 1993; Kaput & West, 1994; Lamon, 1999). I interpret students' additive erring not as execution ‘bugs’

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that need local ‘debugging’ but as diagnostic of students’ lack of useful mental structures for addressing ratio-and-proportion problem situations.

Educators’ sensitivity to students’ experiential resources may put a design ‘spin’ on students’ errors. Specifically, students’ additive proclivities within a multiplicative domain should and could be mined rather than undermined (see Smith, diSessa, & Roschelle, 1993, on misconceptions), given appropriate supporting representations and activities. In particular, close listening to students’ intuitive strategies (Confrey, 1995) can inform the design of teaching/learning materials and activities that foster the development of coherent mathematical constructs (Fuson, 2001; Lampert, 2001). Within the domain of multiplicative mathematical constructs, domain analyses suggest that the conceptual and operative relations of ratio and proportion to other multiplicative structures, e.g., multiplication, division, and fractions, should be revealed to students and utilized in their instruction (Kieren, 1992; Vergnaud, 1994; Confrey, 1998).

The objective of the intervention was to afford and foster mental structures for addressing ratio-and-proportion problem situations. These structures coalesce as synergies of students’ experiential resources (real-world situations, such as constant-rate saving of money) and formal tools that can be seen as framing these experiences in mathematical language (external representations, e.g., the multiplication table). The hope was to anticipate and preempt students’ additive errors entirely by representing ratio-and-proportion situations in familiar spatial-numerical structures that are explicitly multiplicative, or ‘additive-multiplicative’ (see Abrahamson, 2002) and not just additive, such as the multiplication table.

Theoretical Framework and Design Rationale

Mathematics is a rich web of artifacts embedded in cultural practice (Cole, 1996; Ernest, 1988; Schlieman & Carraher, 1992), and the role of mathematics instruction is to help students develop personal meaning for and facility with these artifacts (Cobb & Bauersfeld, 1995; Vygotsky, 1978). Educators can best nurture such personal meaning by helping students relate from their own experiences towards academic structures that capture and re-express these experiences formally (Collins, 1990; Collins & Ferguson, 1993; Freudenthal, 1980; Fuson, Smith, & Lo Cicero, 1997). However, understanding and describing students' would-be 'mathematical' experiences towards designing compatible instructional methods is challenging because students' experiences are phenomenological (Heidegger, 1927/1962) and not conceptual—experiences become cases of mathematical concepts only once they are re-interpreted as such through instructional guidance.

Learning a mathematical concept involves modeling, attending to, and identifying specific quantitative aspects of phenomenologically diverse experiences that help relate these experiences as a single equivalence class (Uttal & DeLoache, 1997; Verschaffel, Greer, & De Corte, 2000; Resnick & Wilensky, 1998). For instance, the phenomenology of Robin, who saves up \$3 per day and that of a petunia that grows 3 mm per day become related as two cases of a $3/1$ rate of increment (or rate of change) only once each experience is described in such terms. Commonality of description that supports insight into such 'family resemblance' (Noble, Nemirovsky, Wright, & Tierney, 2001) or 'conservation' (Piaget & Inhelder, 1966/1969) can involve consistent narrative frameworks (the same 'skeletal story,' Schank, 1990) and uniform symbolic notations

and formats (Vergnaud, 1994). For instance, both the Robin and the petunia phenomena can be related, on the one hand, in terms of a ‘rate story,’ e.g., “Every day, [agent] accumulates [number] [units],” and on the other hand, in terms of a rate table (3, 6, 9, 12, 15, etc., see Figure 1), whether embedded as a column within the multiplication table, or as standing alone. At the same time, just as two diverse experiences become ‘conserved’ through a single skeletal math-story and its related spatial-numeric representation, so, reciprocally, can the interpretation of two different numerical representations become related through a similar story structure. For instance, once a student has related the 3-column of the multiplication table to Robin’s \$3-per-Day story, another contextually-identical story, Tim’s \$5-per-Day saving story, mediates an interpretation of the 5-column of the multiplication table that is consistent with the interpretive schema of the 3-column.

A design that attempts to foster links between real-world phenomena and mathematical representations must consider students’ cognition of the mathematical building blocks underlying the focal concepts. Specifically, because ratio and proportion are multiplicative concepts, it is worthwhile examining students’ cognition of multiplication and division. Students’ intuitive interpretations of arithmetic operations coming from their experiences, e.g., thinking of multiplication as repeated addition and of division as ‘partitive’ (Fischbein, Deri, Nello, & Marino, 1985), underlie and affect students’ comfort with, assimilation of, and ultimate success with implementing concepts that involve these operations. That is, I assumed that students will be more facile with the concepts of ratio and proportion if the design builds on a repeated-addition interpretation of multiplication and a partitive interpretation of division.

Method

Two fifth-grade classrooms (n=19, n=20) with different teachers, in an urban/suburban school each participated in a 10-day instructional intervention that spanned 3 weeks. The author co-taught the unit in the first class and acted mostly as a consulting participant observer in the second class. The author also held briefing and debriefing sessions with the teachers and shared with them daily input from the design team. Thus, the unit was daily modified and ultimately co-constituted by students, teachers, and design team.

Classroom activity alternated between individual and group work on numerical and word problems and class discussions in which students shared their developing strategies. Students received homework and occasionally prepared journal entries. The unit began by exploring spatial-numerical patterns in the multiplication table (rows, columns, diagonals, etc.) and then focused on individual columns in the multiplication table as expressing cumulative rate stories. For instance the 3-column (3, 6, 9, 12, etc.) expressed how many dollars Robin had in her kitty-bank when she began from an empty bank and then put in \$3 per day (see bottom of Figure 1). Similarly, Tim saved money into his doggy-bank at a rate of \$5 per day. Robin and Tim's rate stories were then linked and represented as a 3:5 ratio story (Figure 1 on the right). Selected rows from a ratio story form a 'proportion quartet' (Figure 1 on top; see also Confrey's 'ratio box, 1995). An interpretation of a proportion quartet as lodged in and coming from the multiplication table (Figure 1) explains the structure of the proportion quartet and in particular the factors (or column numbers and row numbers e.g., 2, 7, 3, 5) of the proportion-quartet products (e.g., 6, 10, 21, 35). If one of these products is unknown, as

in missing-value ratio-and-proportion word problems, it can readily be obtained as a reconstruction puzzle solution. The proportion quartet as an abbreviated ratio table retains the situational source of a word problem, and as a mini multiplication table retains computational strategies for obtaining the solution to the word problem.

Data Sources

Students' understanding of the content was evaluated by their performance in pre/post tests that consisted primarily of missing-value ratio-and-proportion word problems from previous studies and by analyzing students' spoken and written explanations to their solutions. In addition, several selected students of diverse mathematical skills were interviewed daily in tutoring sessions. All class and individual work was videotaped and select episodes were transcribed.

Results and Discussion

Generally, students' posttests scores manifested significant progress from the pretests and were as high and at times higher than those of 6th and 8th-grade students in previous studies. Students found the stories engaging and memorable, and were able to generate their own original stories to demonstrate the logical-quantitative relations of rate, ratio, and proportion.

Ultimately, it seems sensible to tap and frame mathematically individuals' real-world experiences as a source of domain-specific structural coherency, because the relatedness of the domain's mathematical constructs evolved over the millennia through human striving for functional efficiency, and this relatedness is expressed and embedded in isomorphic mathematical artifacts, such as the multiplication table. And yet, the appreciation and use of the situation-representation isomorphism depends on first

developing mental structures for adroitly maneuvering within the situation-representation reciprocity. Adroitness in maneuvering within such reciprocity may subsume what we usually term ‘conceptual understanding.’

Behr et al. (1993) describe mathematical meaning as arising through the act of translating between isomorphic representations, e.g., between $1/2$ and 0.5 . Whereas the notion of ‘isomorphism’ may seem an inflated explanatory construct for describing the act of translating between different symbolical notations, e.g., $1/2$ and 0.5 , it appears warranted for describing a translation between symbolical and spatial representations, e.g., between ‘ $1/2$ ’ and a half-shaded rectangle. This is because a geometrical shape can be ambiguous—it does not feel as intrinsically “mathematical” as does an explicitly mathematical symbolical notation, such as ‘ $1/2$,’ and so a student’s appreciation of the isomorphism between the symbol and the shape is contextually contingent and demands intelligent insight. Context is implicitly prevalent in every cognitive act (Light & Butterworth, 1992). But ‘context’ is itself not an explicit, absolute, and universal construct: Context, too, is phenomenologically ambiguous. It, too, has to be interpreted, and this interpretation develops through guidance. I believe that a student’s domain-specific effective interpretation of ‘context’ is precisely what we mean when we speak of conceptual understanding. Translation between representations manifests understanding because it demonstrates an individual’s awareness of the contextual reciprocity between the media of representation.

An awareness of contextual reciprocity between media of representation is mediated by mental constructs (‘symbolic schemas,’ Sherin, 2001) that interpret and generate contextually-dependent translations between representations and situations. If

understanding is the ability to translate between representations (Behr et al, 1993) and we subscribe to the tenets of constructivist instruction that builds on students' experiences (Piaget & Inhelder, 1966/1969; Freudenthal, 1980; Cobb & Bauersfeld, 1995), and if a common mental structure underlies the phenomenology of both real-world situations and spatial-numerical symbolic representations, then the didactic status of students' experiential resources should be reconsidered. Experiences are not just transitory instructional scaffolds that can be discarded once they have mediated formal mathematical meaning. Neither situations nor representations are epistemologically superior or more 'abstract' (Wilensky, 1991). They are each equally important in developing interpretive symbolic schemas that make mathematics meaningful. Thus, situations can enhance rather than impede students' mathematical development if they are fluently related to mathematical representations.

Implications for Mathematics Education

Instructional units in mathematics need not be designed so as to move "up" or "away" from situations. Rather, situations and symbolic representations should be continuously experienced and related as reciprocal resources for the development of mental constructs that enable students to interpret both the situations and the symbols in increasingly sophisticated mathematical ways.

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Solution to:
 Everyday Robin and Tim
 each save at a constant
 rate. If on a certain day
 Robin had \$6 and Tim had
 \$10, then how much did
 Tim have when Robin had
 saved \$21?

	<u>Robin</u>	<u>Tim</u>	
	3	5	
2	6	10	2
7	21	35	7
	3	5	

Proportion Quartet (PQ)
factored outside

PQ as mini MT

PQ as selected
 RT rows (here,
 Rows 2 & 7)

1	2	3	4	5	6	7	8	9	10
1	4	6	8	10	12	14	16	18	20
2	6	9	12	15	18	21	24	27	30
3	8	12	16	20	24	28	32	36	40
4	10	15	20	25	30	35	40	45	50
5	12	18	24	30	36	42	48	54	60
6	14	21	28	35	42	49	56	63	70
7	16	24	32	40	48	56	64	72	80
8	18	27	36	45	54	63	72	81	90
9	20	30	40	50	60	70	80	90	100

Multiplication Table (MT)



	3	5
0	0	0
1	3	5
2	6	10
3	9	15
4	12	20
5	15	25
6	18	30
7	21	35
8	24	40
9	27	45
10	30	50

Every
 day,
 Robin
 puts \$3
 into her
 kitty
 bank,
 and Tim
 puts \$5
 into his
 doggy
 bank

Ratio Table (RT)

Coming from a situation

Rate as single
 MT-column

Every
 day,
 Robin
 puts
 \$3 into
 her
 kitty
 bank

	3
0	0
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24
9	27
10	30

Rate

Coming from a situation
 (The 3-column from MT)

RT as two
 simultaneous
 Rate-columns

Design framework for conceptualizing ratio and proportion. From left and anticlockwise:
 (a) The multiplication table (MT); (b) rate table; (c) ratio table (RT); and (d) proportion
 quartet (PQ). Products and cells of a specific example problem (top left) are enhanced
 here for demonstration.