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THE ODDS OF UNDERSTANDING THE LAW OF LARGE NUMBERS: A DESIGN FOR GROUNDING INTUITIVE PROBABILITY IN COMBINATORIAL ANALYSIS

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Twenty-eight Grade 4 – 6 students participated in 1 hr. clinical interviews in a design-based study that investigated: (a) probability-related intuitions; (b) the affordances of a set of innovative mixed-media learning tools for articulating these intuitions; and (c) the utility of the learning-axes-and-bridging-tools *framework supporting diagnosis, design, and data-analysis. Students intuited qualitative predictions of mean and variance, yet only through grounding computer-based simulations of probability experiments in* discrete–scalar, *non-uniform, multiplicative transformations on a special combinatorial space, the* combinations tower, *could students articulate their intuitions. We focus on a key learning axis, students' confusing likelihoods of unique events with those of classes of events.*

INTRODUCTION

Objectives

This paper reports on a design-based study in mathematics education. The study was designed to advance three interdependent lines of research: (a) *theory of learning*—probing late-elementary and middle-school students' intuition for probability; (b) *design*—examining the roles a set of innovative learning tools may play in supporting students' articulation of any probability-related intuitions they may have; and (c) *design theory*—evaluating and improving a framework for mathematics education that guides a researcher from diagnosing a learning problem through to design and to data analysis.[1] The mathematical domain of probability was chosen as particularly auspicious for studying student articulation of intuitive interaction with specialized tools due to: (a) an "intuition gulf" created by this domain's ambiguous treatment of 'prediction'—the indeterminism of the individual sampling action as compared to the by-and-large predictability of aggregated results of sufficiently numerous sampling actions; (b) students' deep rooted and lingering confusion and even superstition that such ambiguity entails and the detrimental effect of these confusions on problem solving; and (c) the roles learning tools could play in enabling students to confront this ambiguity and reconcile it in the form of coordinated conceptual understanding (Abrahamson & Wilensky, 2004a; Liu & Thompson, 2002; Stohl Drier, 2000; Wilensky, 1997).

The study reported in this paper was the first stage of a larger design-based research project that includes: (1) one-to-one clinical interviews; (2) focus-group

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studies; and (3) classroom interventions. The objective of the interviews was to elicit students' *learning issues* (Fuson & Abrahamson, 2005) in this design, that is, potentially problematic aspects of the targeted mathematical content (probability) as embedded in, and emerging from, student interactions with the designed materials. In particular, we investigated whether students could build a coherent, if largely qualitative, account of the Law of Large Numbers—an account that recruits students' relevant, yet possibly conflicting, intuitions and bridges these intuitions. By 'intuition' we refer broadly to mental/perceptual action models that students *tacitly* bring to bear to interpret situations in the context of a mathematical problem (Fischbein, Deri, Nello, & Marino, 1985; Lakoff & Nuñez, 2000; Polanyi, 1967).

Theoretical Perspectives

Three related theoretical perspectives informed the design of this study, including its materials, procedure, and data analysis. One perspective, *learning axes and bridging tools* (Abrahamson & Wilensky, 2004b) characterizes learning as tackling and resolving pairs of conflicting ideas residing along conceptual "axes" [plural of 'axis']. Based on a domain analysis, a designer taking on a design problem can depict students' difficulty in terms of a lack of opportunities for addressing a set of concept-specific learning axes. These axes then frame a design plan. Once the axes are vested in actual tools, one speaks of 'learning issues,' i.e., the pragmatics of constructing new conceptual understandings within a particular design. Thus, the learning materials and activities are designed to embody, foreground, and juxtapose the axes to enable students to resolve the conflicts. Such juxtaposition is enhanced by embedding both conflicting ideas within a single ambiguous object. The second perspective, the *apprehending zone,* is that students learn through building connections within and between situations and mathematical objects—teachers model problem solving to facilitate the building of these connections (Fuson & Abrahamson, 2005). The third perspective positions mathematical objects as more than arbitrary epiphenomena aiding mathematical reasoning. Rather, conceptual knowledge may be embodied in learners' growing relations with artifacts supporting the construction of understanding (Gigerenzer, 1994; Pirie & Kieren, 1994; Vygotsky, 1978). Together, these perspectives suggest the criticality of the craft of design: Mathematical objects could be more than learning 'supports'—they could become internalized as permanent and inextricable imagistic vehicles of mathematical reasoning.

A key domain-analysis principle implemented in the design is that reasoning about probability from a complementarity of 'macro' and 'micro' perspectives is critical for deep and 'connected' understanding of the domain (Wilensky, 1997).

METHOD

In 9 visits spanning 3 weeks, 28 Grades 4-6 students from a K-8 suburban private school (33% on financial aid; 10% minority students) each participated in a one-to-one semi-clinical interview (Ginsberg, 1997; mean 56 min., SD 12 min.). From the pool of volunteering students, we selected students representing the range of mathematical performance in their grade levels as reported by their teachers. Also, we balanced for gender. All sessions were videotaped for data analysis.

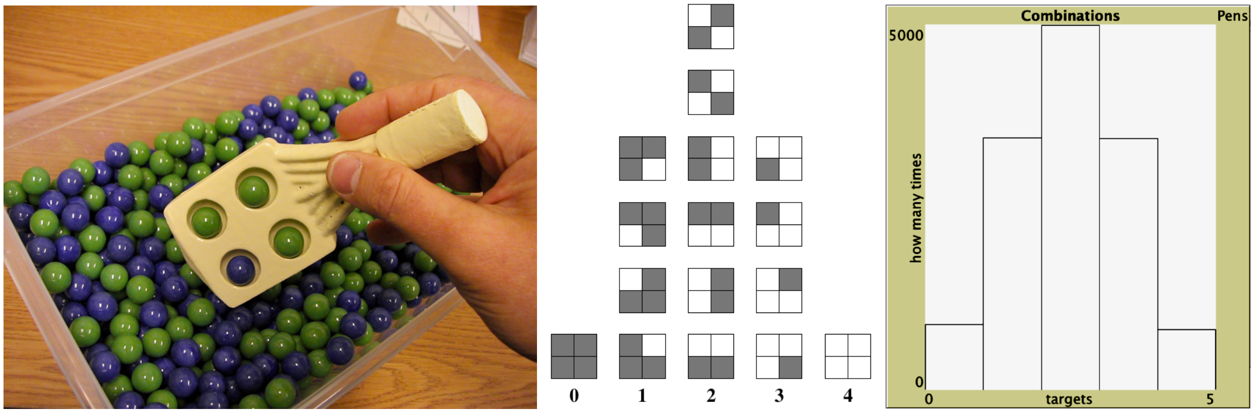

Figure 1. Selected materials: marble scooper, combinations tower, and 4-Blocks.

The materials were all embodiments of the *4-block* stochastic device from the ProbLab experimental unit (Abrahamson & Wilensky, 2002, 2005), implemented in different media (see Figure 1, above). The *marble scooper* device scoops a fixed number of marbles out of a vessel containing many, e.g., 100 green and 100 blue. The *combinations tower* is the combinatorial space of all 16 possible 4-blocks arranged by the number of light-colored squares in each. Students use crayons to build the tower from paper cards, each featuring an empty 4-block grid. Two simulations built in NetLogo, a multi-agent modeling-and-simulation environment (Wilensky, 1999), included: *4-Blocks*, where four squares independently choose randomly between two colors—the program dynamically records the blocks by number of green squares; and *4-Blocks Stalagmite* (see Figure 2, below), in which 4-blocks are generated randomly, yet *the samples themselves* are stacked in a pictograph bar chart. The simulation can be run under various conditions, e.g., rejecting repeat samples (Figure 2a) or keeping them (2b).

Using microgenetic analysis (Siegler & Crowley, 1991), we studied the data to characterize properties of the learning tools, activities, and prompts that enabled students' to move from difficulty to understanding during the interview, where 'understanding' was operationalized as students' manifesting stable and coherent discourse about new content in terms of the tools and connections between them.

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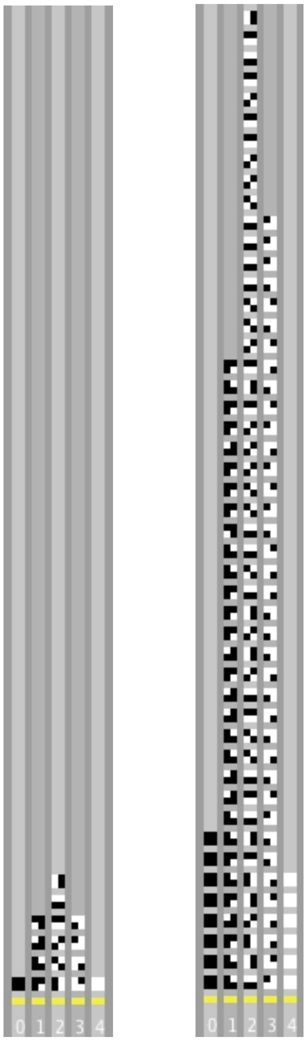


2a. *2b.*

Figure 2. Two appearances of a 4-block "stalagmite."

RESULTS AND DISCUSSION

Intuitive Judgements and Strategies

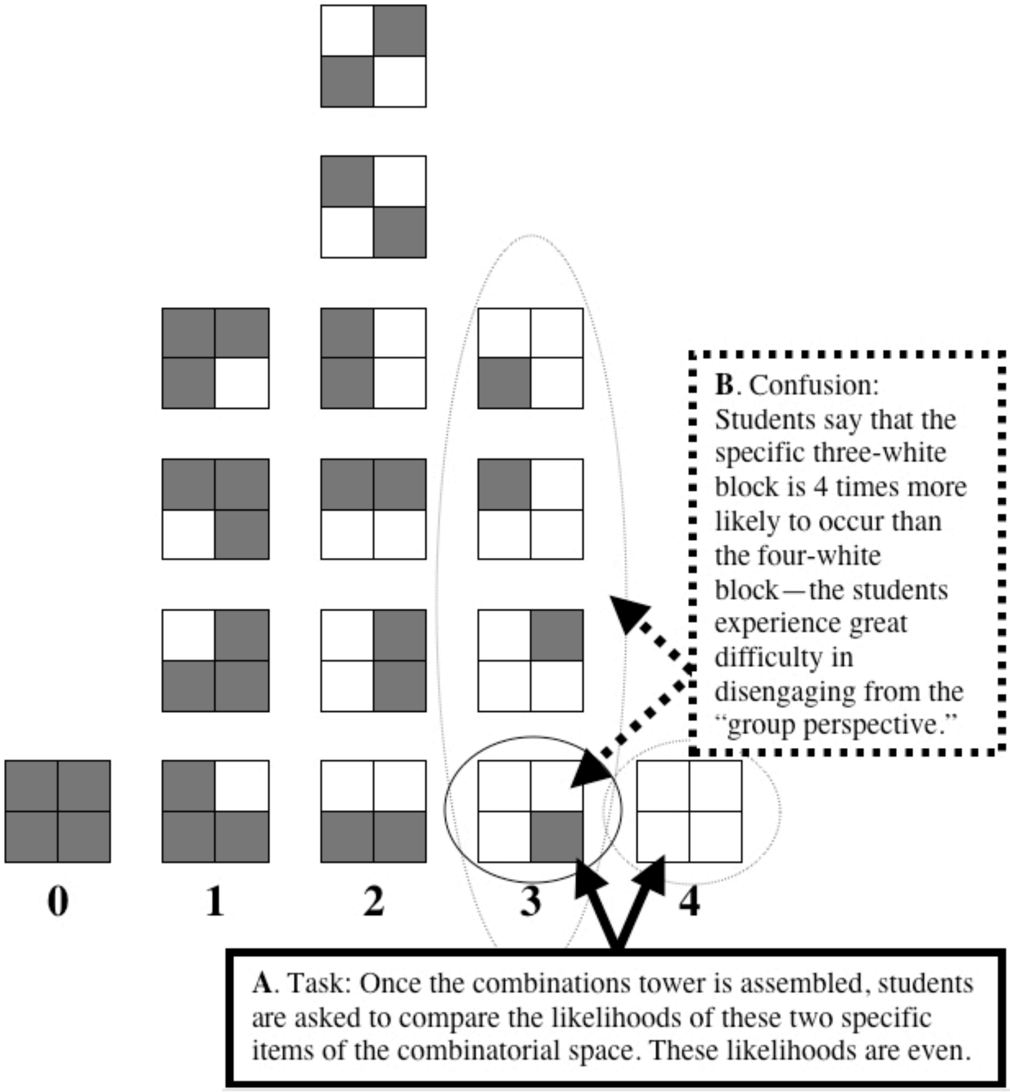
All but two students initially predicted a 2-green/2-blue 4-block as the most common scoop from the bin. Asked to support this guess, most students said, "It just seems so," pointing to the apparently equal numbers of green/blue marbles in the bin ("half–half"). Others said they had "picked this [idea] up" or that they did not know. Not a single student initiated an exploration of the combinatorial space as a means to warrant or validate that intuition. Even following prompts to construct the combinatorial space, all students asked whether they should include the permutations. Typically, the event *classes*, e.g., patterns with exactly 1 green square, emerged only through actions of generating different patterns and assembling the combinations tower. For example, a student who had created only one pattern with a single green square realized that this square could be located in each of the four 4-block locations. Thus, attending to event classes emerged as a pragmatic principle for mathematical inquiry.

Drawing a Compound Event From a Hat

Students understood that each 4-block outcome resulted from the concurrence of 4 independent random outcomes. Yet, in referring both to the combinatorial space (the combinations tower) and to outcome distributions from repeated sampling (the simulated experiments), students came to treat the 16 possible compound events as though these were 16 independent events of equal likelihood—as though, for example, the experiment were analogous to drawing one of the 16 cards from a hat. Such "clumping" of compound events, supported by the combinatorial-analysis format (the cards), appears to have enabled students to reflexively apply to the set of compound events their intuition for a set of independent events. Albeit, this recursive strategy does not easily apply for $p \neq .5$.

Learning Axes: "My Mind Keeps Going Back and Forth"

The learning axis 'specific event vs. event class' (see Figure 3, below), which we perceive as key to this design (see next section), posed great difficulty for several Grade 6 'High' students, who required a mean of 9 min. until stability.



B. Confusion: Students say that the specific three-white block is 4 times more likely to occur than the four-white block—the students experience great difficulty in disengaging from the "group perspective."

A. Task: Once the combinations tower is assembled, students are asked to compare the likelihoods of these two specific items of the combinatorial space. These likelihoods are even.

Figure 3. Students' difficulty with the learning axis 'specific event vs. event class.'

Students' Discrete–Scalar Insight Into the Law of Large Numbers

Students understood that event classes have different likelihoods according to their relative sizes, with larger event classes having "better odds" in the experiments, e.g., it is more likely to randomly get any two-green 4-block than any three-green 4-block, because there are six items in the former group and only four in the latter. Although only two 'High' students could express these *inter*-class relations multiplicatively, students learned to use these relations to express what we have termed a *discrete–scalar* multiplicative model of the outcome distribution.

The traditional representation of binomial distribution (see Figure 4, below, on the left) can be interpreted as an ambiguous figure enfolding theoretical and empirical constructs. These can be bridged by an *itemized distribution*, i.e., the combinations tower. This bridging tool illuminates stochasm as theoretical-to-empirical *transformation*, as follows. All events have equal opportunities to be repeated in the experiment, so outcome categories holding a larger variety of different events collect more groups of repeated events and therefore collect a larger total of outcomes. For instance, if each of the 16 different possible events appears in the experiment about 3 times, then a column collecting 4 types of outcomes (e.g., see Figure 4, below, on the right) accumulates a mixture of 4 sets, each of about 3 outcomes, for a total of about 12 outcomes (see also Figure 2).

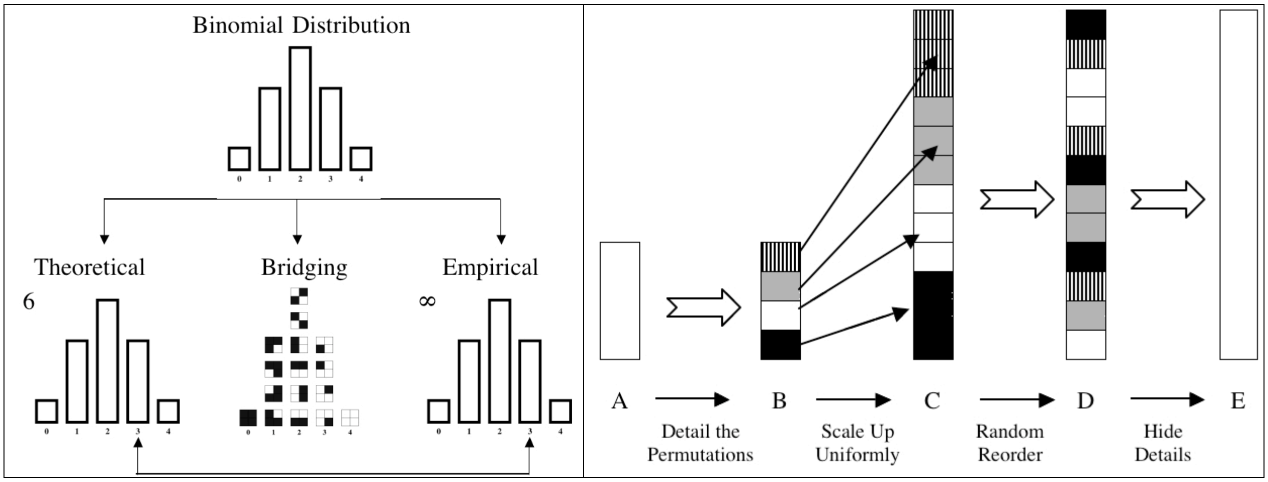


Figure 4. Bridging theoretical and empirical aspects of binomial distribution.

Students could investigate this line of reasoning within the 4-Block Stalagmite simulation, because, unlike in traditional representations, the samples *themselves* were all stacked in the columns (see Figure 2). Further, students could readily compare the outcome distribution and the combinatorial space, because these two structures were designed to appear similar (see Figure 1). We envisage such understanding as supporting a treatment of the sequence of column heights, e.g., 1-4-6-4-1, as a set of multipliers, i.e., as coefficients in the binomial function; the multiplicand 'scalar unit' would be the events' mean number of occurrences.

CONCLUSIONS

Design

The design was an example of a framework by which the designer first identifies a learning axis, then actuates this axis in the form of objects and activities that embody this axis as a learning issue; students must confront this issue and unravel it toward deep understanding of mathematical content. The demonstrated ubiquity of students' probability-related intuition together with the effectiveness of this design in enabling students to investigate this intuition suggest that this design could potentially be developed into a unit used in late elementary school and certainly in middle school. Whereas basic fluency with rational-number models appears to have helped a couple of Grade 6 students perceive the outcome distribution as a proportionally-equivalent scaling-up of the combinatorial space, such fluency is not necessary for appreciating the scalar–discrete and stochastic transformations explored in our design. Finally, in future development of the design we will explore its potential extension to cases in which $p \neq .5$ and also study its connections to normative, symbol-based, mathematical expressions.

Building on Intuition: "It's What I Was Trying to Say But Didn't Know How"

Just how intuition is grounded in teaching–learning contexts is a difficult yet important question to pursue, because not all students are able to recognize a resonance of their intuition within these contexts. Moreover, just because the combinations tower enabled students to *validate* and possibly *ground* their intuitive judgment, it does not necessarily follow that this validation was an *articulation* of the intuition itself. The intuitive judgment needn't have been combinatorial—it may have been some manner of proportional reduction or mapping of the marble population onto the marble-scooper 4-block template. This should be researched.

Teaching

Focusing classroom discussion on the learning axes is challenging, because students may hold fast onto their confusions. Nevertheless, intuitions should be recruited into learning spaces even if these intuitions initially appear vague or misleading, because they will persist anyway—students will achieve a sense of understanding only if the learning issues are faced head-on by probing and discussing interpretations of objects within problem-solving activity contexts.

This study has contributed an innovative design. The design framework outlined herein, too, could help education practitioners, both in the domain of probability and beyond. Finally, we call for further research on the nature of intuition and how it may be sustained through to deep mathematical understanding.

NOTE AND ACKNOWLEDGMENTS

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REFERENCES

Abrahamson, D., & Wilensky, U. (2002). *ProbLab*. Evanston, IL: The Center for Connected Learning and Computer-Based Modeling.

Abrahamson, D., & Wilensky, U. (2004a). ProbLab: A computer-supported unit in probability and statistics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Annual Meeting of the International Group for the Psychology of Mathematics Education Vol. 1* (pp. 369). Bergen, Norway: PME.

Abrahamson, D., & Wilensky, U. (2004b). *S.A.M.P.L.E.R.: Statistics As Multi-Participant Learning-Environment Resource*. Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.

Abrahamson, D., & Wilensky, U. (2005). Understanding chance: From student voice to learning supports in a design experiment in the domain of probability. In G. M. Lloyd, M. Wilson, J. L. M. Wilkins & S. L. Behm (Eds.), *27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Roanoke, VA: PME-NA.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Fuson, K. C., & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase problem-solving models. In J. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 213-234). New York: Psychology Press.

Gigerenzer, G. (1994). Where do new ideas come from? In M. A. Boden (Ed.), *Dimensions of creativity* (pp. 53-74). Cambridge, MA: M.I.T. Press.

Ginsberg, H. P. (1997). *Entering the child's mind*. New York: Cambridge University Press.

Lakoff, G., & Nuñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*: Basic Books.

Liu, Y., & Thompson, P. (2002). Randomness: Rethinking the foundation of probability. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant & K. Nooney (Eds.), *Proceedings of the Twenty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA* (Vol. 3, pp. 1331–1334). Columbus, OH: Eric Clearing House.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(165-190).

Polanyi, M. (1967). *The tacit dimension*. London: Routledge & Kegan Paul Ltd.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: a direct means for studying cognitive development. *American Psychologist, 46*(6), 606-620.

Stohl Drier, H. (2000). Children's meaning-making activity with dynamic multiple representations in a probability microworld. In M. L. Fernandez (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for Psychology of Mathematics Education*. Columbus, OH: Eric Clearing House.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Wilensky, U. (1997). What is normal anyway?: Therapy for epistemological anxiety. *Educational Studies in Mathematics, 33*(2), 171-202.

Wilensky, U. (1999). NetLogo. The Center for Connected Learning and Computer-Based Modeling, Northwestern University, Evanston, IL.