Fostering the Emergence of an Embodied Cognitive Artifact: The Case of the Number Line in a Design for Probability

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I trace the emergence of a mathematical instrument, the number line, in the context of student engagement with a situated-probability problem-solving interview task involving manipulatable objects. I argue that consequent semiotic–ontological ambiguity engendered struggle with generative conceptual confusion. Namely, as students conducted combinatorial analysis to create the sample space of a random generator, the objects they built to express the stochastic *events* served both as tickmarks on an emergent number line and as *outcomes*, members of those marked events. Negotiating the tickmark-vs.-member semiotic ambiguity challenged, then facilitated the dyad's discourse over the event-vs.-outcome learning axis, which I have implicated as key to deep understanding of the binomial. Extrapolating from the data, I examine the phylogenesis-recapitulates-microgenesis conjecture (a reversal on Haeckel) by which the historical evolution of the number line may have proceeded from objects to inscription; I draw an explorative implication that mathematics instruction could follow suit, i.e., that students could re-invent the number line as an inscribed ordinal sequence of sorted objects.

Objective and Background

This paper departs from the observation that some mathematics representational systems such as the number line evolved in disparate cultures and are in ubiquitous use where mathematical literacy, practices, and discourse are valued. I offer the conjecture that these cross-context similarities in form and function are non-arbitrary and reflect commonalities of our physical, cognitive, and perceptual makeup as well as shared features of our cultural contexts of practice, such as industry and commerce.[1] To evaluate this conjecture, I present and examine empirical data of cases in which, I argue, the number line evolved in the "micro-culture" of an interviewer–student dyad engaged in situated mathematical discourse over the "micro-historical" duration of a clinical interview. Building on the related traditions of cognitive development and cultural phylogenesis (Saxe & Esmonde, 2005), cultural semiotics (Lemke, 1998, 2002; Radford, 2003, 2006), distributed cognition and conceptual blending in professional practice

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[1] The pair 'form and function' is borrowed directly from G. Saxe and collaborators. I will sometimes use '*structure'* that has a similar sense to 'form' yet foregrounds the critical roles of substantive materials underlying emergent systems of practice and also suggests an affinity of mathematical objects with its cognates 'con*struct*ion' and 'in*struct*ion.'

(Fauconnier & Turner, 2002; Hutchins, 1995, 2005; Hutchins & Palen, 1997), embodied cognition and gesture studies (Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999; Barsalou, 2008; Roth & Welzel, 2001), learning sciences (Collins & Ferguson, 1993), and anthropologically oriented discourse analysis (Goodwin, 1994; Stevens & Hall, 1998), I demonstrate how a canonical instrument such as the number line may emerge serendipitously from the mediated interaction of epistemic and material resources available to students as vehicles of reasoning, documentation, and communication within situated problem-solving activity contexts. Moreover, I argue that the emergence of mathematical artifacts serves pedagogical objectives and should therefore be analyzed, understood, and fostered. In particular, I show how artifacts and knowledge are co-constructed—they co-emerge dynamically, reflexively, such that students can develop deep understanding of mathematical content as they build materials and discover and reflect on their meaning. Fostering such productive engagement requires a principled framework that articulates paths for authentic exploration within the constraints of carefully planned concept-targeted design.

Designing Cognitively Ergonomic Learning Tools Supporting Students' Progress From Intuition to Inscription: Mathematical Reasoning Viewed as Semiotic Process

To begin a modest exploration of the plausibility of the 'mathematics-artifact emergence' conjecture, I calibrate my analytic lens to microgenetic resolution and closely examine brief interactions of individual students operating in a learning environment created in the context of an empirical investigation of mathematical cognition. Namely, I analyze young learners' spontaneous construction of mathematical representations in studies of children's intuitive inferences and quantitative reasoning. In those studies, late-elementary and middle-school participants in clinical interviews performed perceptual judgments of quantitative relations embedded in visual stimuli, drew intuitive inferences based on those judgments, and then used a range of media in attempt to warrant those inferences in accord with the logico–mathematical argumentation forms of rhetoric assumed as privileged in that discursive context. The interactions thus took on a semiotic register in which mathematical formulations progressively emanated at the dynamic conjunction of *presymbolic notions* and available *semiotic means of objectification* (see Radford, 2003, for the semiotic framing of mathematical learning). One such formulation, I will argue, was the number line, which "came to be" epiphenomenally to a chain of authentic, mundane physical activities of constructing and sorting a set of objects. Specifically, I will demonstrate how the number line emerged, within a mandated activity sequence, as a *cognitive artifact* (Norman, 1991) that subsequently served students in distributing across space, time, and agents a complex procedure they were first learning to perform as well as in anchoring arguments over properties of the mathematical objects and their meaning within the problem-solving context.

The empirical context within which I am exploring for semiotic roots of embodied artifacts is a research project, *Seeing Chance* (Abrahamson, PI), that is looking into young students' cognitive resources relevant to the mathematical concept of probability as well as at the potential roles that mixed-media technology can play in fostering opportunities and framing viable trajectories for students to build on their resources in developing a grounded fluency with disciplinary tools of practice such as combinatorial analysis (Abrahamson & Cendak, 2006).

I have described my design work as the process of developing *cognitively ergonomic learning tools*—artifacts attuned to the tacit somatics of mathematical reasoning and geared to support students' problem-solving-based mental composition of target content en route *from intuition to inscription* (Abrahamson, 2007a, 2007b). Specifically, I have outlined and empirically evaluated a principled framework for implementing constructivist/constructionist pedagogical philosophy in the form of actual concept-oriented learning materials, activities, and facilitation guidelines (Abrahamson, 2006b, 2006c; Abrahamson & Wilensky, 2007). Key to the designer's practice within this framework is an analytic decomposition of mathematical representations into binary conceptual elements and creation of opportunities for students to generate these elements and then negotiate, and reconcile the elements and, so doing, reconstruct the target concept. In Abrahamson (under-revision) I decompose the binomial distribution into event-based intuition and outcome-based analysis. Through engaging in combinatorial analysis, students confront an emergent tension between their event-based and outcome-based orientations toward the sample space. This process as well as the supporting role that the number line implicitly plays in the process will be explained below in the design section and then through the data analysis.

Design-Based Research: Framing a Parallel Pursuit for the Embodied Roots of Probabilistic Cognition and Learning Materials for Students to Build on These Roots

The Seeing Chance project is conducted in the design-based research approach (Barab et al., 2007; Brown, 1992; Collins, 1992; Confrey, 2005; diSessa & Cobb, 2004; Edelson, 2002), in which education-related phenomena are studied through implementing a didactical intervention that creates empirical context to evaluate the researchers' conjecture regarding cognitive, social, and/or other mechanisms that factor into the phenomena in question. My conjecture, which builds on previous research into natural stochastics (Gigerenzer, 1998; Wilensky, 1997; Zhu & Gigerenzer, 2006), has been that students' probabilistic intuitions—often treated as misconceived (Fischbein & Schnarch, 1997; Konold, 1989; Tversky & Kahneman, 1974)—are highly sensitive to contexts of enactment and in particular to the media and representational systems used in the elicitation and measurement of the intuitions. Indeed, given suitable contexts, students who have not studied probability can nevertheless generate assertions that are aligned with normative mathematical knowledge. In particular, students as young as Grade 4 have correctly anticipated, if qualitatively, the empirical outcome distribution of a random generator and then learned to synthesize inferences from their intuitive judgment with inferences from the sample space of this random generator (Abrahamson & Cendak, 2006; see Schön, 1981, on 'synthesis' as a principle and process in constructivist design).

As I iteratively tune my design (the actual learning materials, whether traditional or computer-based) to students' apparent intuitive resources that I elicit and analyze from study to study, I theorize students' modes of appropriation of mathematical concepts as embodied negotiation between routinized theorems-in-action, which have become second nature prior to the intervention through repeated use, and suggested modes of engagement with new mathematical objects that the teacher or design-researcher introduces into the classroom microculture (Abrahamson, 2004; Fuson & Abrahamson, 2005). Thus, the empirical occasions of "test-driving" and improving my design constitute arenas for my own inquiry and improvement of my theoretical models of the nature of mathematical learning. This paper differs somewhat from canonical reports coming from the design–implement–analyze cycle so typical of design-based research, in that here I take pause to reflect on occupational grounds explanatory of universal convergence on standardized mathematical instruments.

Following, I explain the design used in our study, interleaving the explanation with empirical data from implementations of this design (Abrahamson, under-revision; Abrahamson, Bryant, Howison, & Relaford-Doyle, 2008; Abrahamson & Cendak, 2006). Next, after an interim reflection on the diagrammatic elements of the number line, I will discuss data from our study so as to demonstrate how the number-line was woven together from those elements in the context of the problem-solving activity. The paper ends with an extrapolation from this small-scale study to a proposed (bold) account of historical processes underlying the emergence of the number line.

Setting the Context: A Design-Based Research Study of Probabilistic Cognition That Engages Participants in Mediated, Situated Problem-Solving Activities With Innovative Mixed-Media Learning Materials

Outline of the Design: Toward Coordinating Event-Based Intuition and Outcome-Based Analysis

Twenty-eight 4th-6th grade participants in a semi-clinical one-to-one interview engaged in the solution of the following situated problem. A medium-sized plastic tub containing hundreds of green and blue marbles of equal numbers was laid on the student's desk (see Figure 1a, next page). A special utensil designed to draw out exactly four marbles was shown to the participant, and s/he was asked, "What will we get when we scoop?" Typically, these students, who had had no formal instruction of probability content, nevertheless correctly estimated a '2-green, 2-blue' sample as the predicted mode outcome, '4-green' or '4-blue' as the rarest outcomes, and '3-green' or '3-blue' as in-between in their expected outcome frequencies (Abrahamson & Cendak, 2006). Asked to warrant their claim, all students pointed out the 'half–half' green-to-blue marbles ratio (see Abrahamson, 2007c, for an analysis of this judgment from a gesture-studies perspective; see Tversky & Kahneman, 1974, on the 'representative heuristic' by which students are presumably mapping the essence of the random generator onto the sample size). Notably, no student initiated attention to the different possible configurations (permutations) that each of the five events could take.[2]

Next, having estimated the *probable*, participants turned to address the *possible*. Namely, students engaged in combinatorial analysis of the random generator. Specifically, students were guided to use a set of empty 4-block cards (made of stock paper) as well as green and blue crayons to create "all the different scoops we could get." Once students completed building the entire sample-space consisting of 16 unique outcomes, they were guided to assemble it in a histogram-like structure, the *combinations tower* (see Figure 1b). Upon beholding the completed combinations-tower structure, all but one of our

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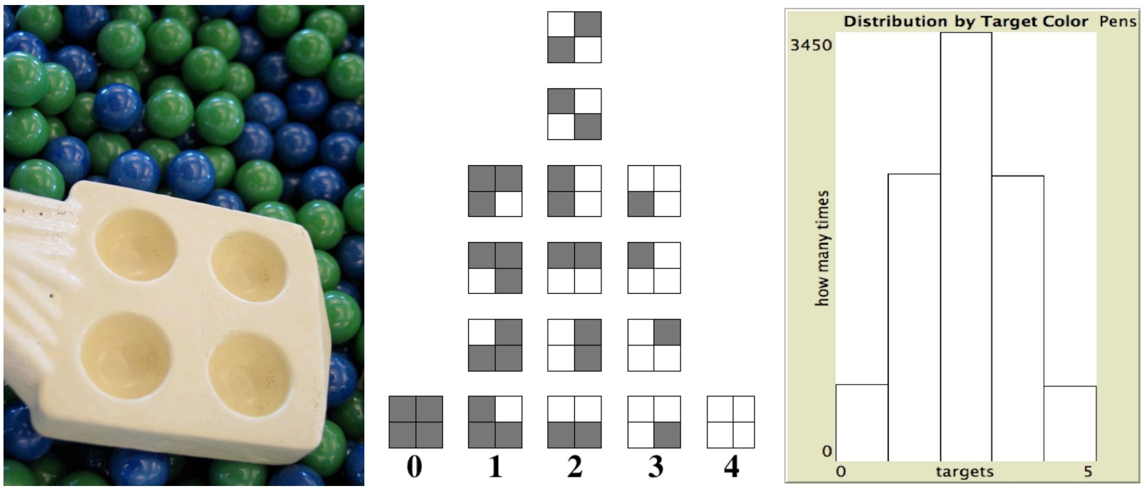
[2] Granted, an expected-value approach would be sufficient to establish a 2-green scoop as the mode, yet only through combinatorial analysis—either manual or equation-based—can the expected outcome distribution be determined, whether qualitative (less, more, most, more, less) or quantitative (1:4:6:4:1).

participants recognized the structure as apparently furnishing a logico–mathematical warrant for their earlier intuitive inference that a 2-green card would be most likely to occur. Namely, students said that there are more cards in the 2-green column than in any other column and therefore a 2-green scoop would happen more than any other scoop (see Stavy & Tirosh, 1996, for a heuristic that could explain our participants' inference). This intriguing moment has been a focus of our critical study of the nature of mathematical concepts, intuition, insight, learning, and design (Abrahamson et al., 2008).



a. *b.* *c.*

Figure 1. Selected learning tools and constructions in the study—theoretical and empirical embodiments of the 4-block mathematical object: (a) The marble scooper; (b) the combinations tower; and (c) an actual experimental outcome distribution produced by a computer-based simulation (see http://edrl.berkeley.edu/design.shtml to operate the interactive modules).

At this point in the interview, the students, who are addressing the question of expected outcomes from the random generator, have triangulated their intuitive inference based on perceptual judgment of the marbles box with their interpretation of the sample space created through a formal process, combinatorial analysis. By and large, students are confident in their assertions and feel they have adequately warranted their initial estimations. Yet, the students have not yet actually conducted the experiment, such that their confidence is grounded in the triangulation of the intuitive and analytic. The intervention ends with computer-based activities, in which students conduct simulated experiments with the marbles-box random generator to witness the dynamic realization of their triangulated prediction and connect it toward the formula of the binomial.[3]

[3] We have explored the viability of an explanatory model by which the actual empirical outcome distribution in the simulated experiments can be seen as a multiplicative–stochastic stretch of the combinations-tower sample space (for details, see Abrahamson, under-revision; Abrahamson & Cendak, 2006). The algebraic content—namely the binomial formula—was created for an older audience as was implemented with undergraduates (Abrahamson, 2007d, under revision).

Students' Difficulty in Differentiating Events and Outcomes: Implications for Mathematical Cognition and Design

Note in Figure 1b the bottom row of the combinations tower that consists of five outcomes. During the combinatorial analysis, the dyad negotiated over whether or not just five cards such as these five—with 0, 1, 2, 3, and 4 green squares, respectively—can be said to exhaust the sample space. Findings were consistent with the earlier phase of the interview: Just as students had ignored the idea of permutations in attempting to warrant their intuitive estimation of the mode scoop, so now the students persevered in their obliviousness to the pertinence of permutations for determining the anticipated outcome distribution. In fact, some students asked explicitly why they should bother attending to the entire sample space if they are attempting to determine properties just of those five cards. Moreover, when we subsequently asked students to compare the likelihoods of two outcomes—a card from the 3-green column and the 4-green card—students by-and-large did not initially appreciate that these cards are in fact equiprobable. In particular—and robustly—students experienced great difficulty in articulating the difference between an event and an outcome—students 'saw' a *particular* card, say the 4-block with a single blue square in its top-right corner, as meaning *any* outcome with exactly one blue square.

In our publications, we have built on analyses of these data to claim that prior studies ostensibly demonstrating humans' fallible probabilistic cognition (Tversky & Kahneman, 1974) were set up such that participants were in fact giving correct answers to different questions. That is, when asked which, if any, of a HHHH or HTHT sequence of four coin flips is more likely, the reason that the vast majority of participants point to the latter as more likely—a mathematically incorrect answer—is that the participants are only counting the numbers of H's and T's. That is, people gazing at 'HTHT' see it as "2H, 2T"—a combination and not one of its particular permutations. People ignore the order of independent outcomes, I contend, because probabilistic intuition is event based, not outcome based.[4] Indeed, an *event* with exactly two H's is more likely than an event with four, whereas these two specific *outcomes* are in fact equiprobable. Yet people by-and-large do not see an outcome qua outcome but, tacitly, qua event. To achieve a deep understanding of probability, we have therefore argued, students need to synthesize their event-based intuitive resources with analytic-based mathematical procedures.

Whereas the data from the full interview have been informing our understanding of aspects of design theory (Abrahamson & White, in press)—not only issues of mathematical cognition and learning—in the current paper I will focus only on the second third of the interview protocol, in which students are guided to construct and assemble the sample space. In particular, I will examine transitions into and out of three milestone moments along the construction phase—5 outcomes, 16 outcomes, and the combinations tower—to investigate how the distributed sample space was both constructed *by* the sequence of coloring-and-assembly actions and, reflexively, *enabled* the critical insight that the sample space (in the form of the combinations tower) indexes the relative frequencies of the five events—the insight that thus first enabled students to

[4] More precisely, 'events' and 'outcomes' are not yet available in intuitive probabilistic judgment as distinctive frames, so that a better, if colloquial, term might be the ambiguous term "things," as in "the things we can get"—i.e., the objects students see.

use quantitative descriptors as correlates of their initially qualitative assertions. What was unique to this process, I submit, is that the 4-block cards, at least the five cards at the base of the combinations tower, constituted both the “tick marks” of an emergent number line and the very objects whose quantitative values were indexed by the number line these cards embodied. This reflexive nature of information that carries forth its own “beat”—information that is both encoded by and encoding of its implicit referential grid—has been identified and discussed in Bamberger’s (in preparation) analyses of action and invented notation in the context of musical experience (and see Lyotard, 1971, on the figural nature of discourse).

Available Media and Emergent Instruments: A Circumspective Deconstruction of Mathematical Artifacts Toward Analyzing Students’ Situated Phenomenology

To evaluate my conjecture that the number line emerged through students’ combinatorial analysis, and especially seeing as this number line is not of canonical form, we must first briefly ask, What was the nature of the available media that they may have enabled the emergence of the number line? Furthermore, what would constitute a number line? What are the necessary structural characteristics of a mathematical artifact such that we may agree on criteria for inspecting its apparent emergence? The exercise of articulating the nuts and bolts of the number line may serve another goal: We would be in a better position to analyze individual students’ understanding and difficulty with this artifact. Such analysis, in turn, would gear us toward engineering pedagogical responses to support students who are struggling to understand the structure and function of this artifact. The following section, thus, examines the requisite representational elements of the 4-block and then turns to examine the corresponding elements of the number line.

Structural Analysis of the 4-Block

Our descriptions thus far have treated the available media as pivotal contributors to the emergence of mathematical understanding. Yet how were these media able to serve such a role? In particular, what—if any—are the structural properties of the 4-block uniquely tuned to the problem-solving context and target content? Let us consider features of 4-block so as to inquire into its use in the specific context of solving the given problem of warranting the intuitive prediction of experimental outcomes. The following list may appear to contain structural minutia, just as the subsequent list of the number line requisites might, yet it is at this level of detail that the nature of students’ activity is determined as they interact dynamically with the available media—a level of scrutiny that is to include students’ planned, inadvertent, and arbitrary micro-actions.

- *Identity*: The sixteen unique 4-block cards differ in their specific configuration of green and blue squares. Yet the material substance of all the cards is identical—the cards are all “values” of the same “variable,” i.e., they are explicitly engineered to portray possible states of the scooper.
- *Shape*:
    - The marbles scooper is square, and the 4-block “inherited” its shape from the scooper. The scooper’s shape was designed so as to match the appearance of another mathematical object that we have used in a related experimental instructional unit for basic statistics, a 2-by-2 sample out of a

“mosaic” of thousands of green and blue squares on a computer screen (Abrahamson, 2006b; Abrahamson, Janusz, & Wilensky, 2006; Abrahamson & Wilensky, 2004a, 2004b, 2005, 2007). The goal of using similar images across contexts is to support students’ development of a coherent *conceptual field* (Vergnaud, 1994) for probability and basic statistics, topics that I regard as akin (Abrahamson, 2006a).

  - The marbles scooper has a flat surface so as to enable the scooped marbles to roll into the four concavities and let the rest of the marbles drain out.

- *Rotational symmetry*: The 4-block, ensconced in its square mold, has four possible states around the rotational axis of symmetry. This means that all sixteen cards are identically shaped regardless of their particular orientation.
- *Linearity*: The 4-block is square and therefore has straight edges in all rotations. The combinations tower is assembled as the concatenation of contiguous 4-blocks. Therefore, the concatenation of 4-blocks creates straight lines. The rotational symmetry makes for both horizontal and vertical linear continuities.
- *Units*: The following qualities: (a) the equal sides of the 4-block; (b) the linearity of its concatenated continuity; (c) the cardinal property of the cards, i.e., the ‘number of green squares’ in each; and (d) the choice to arrange the cards from left to right in accord with the ‘number-of-green’ property, are all coordinated in the form of an emergent scale-like structure running from 0 through to 4. This structure resembles a number line and may function as such, in that consecutive 4-blocks are equidistant and each marks the location (the column) where corresponding data objects can be placed along the vertical axis.
- *Tessellability*: The identity of all 4-blocks, their rotational symmetry, and the linear continuity of their contiguities make an array of 4-blocks a tessellation. At a mundane level, the square shape of the 4-block cards greatly facilitated our preparation of a set of discrete cards as materials for the interview, because the specialists at the copy store could prepare an 8-by-11 inch sheet of paper containing 3-by-4 cards with just five cuts (the customer pays per cut, so this was economical too).
- *Discreteness*: The concreteness of the sixteen cards lent them a dual status: They were collectively both a ‘sample space,’ in the sense that they represented all the possible unique outcomes, but they were also a ‘sampl*ing* space,’ in the sense that randomly selecting one of these sixteen cards is commensurate with scooping from the marbles box.[5] Students’ actions and language suggested that they were responding to the ambiguous role of the cards as both signs and objects.

Having examined structural and functional properties of the 4-block—the key available medium in the combinatorial analysis procedure—we will now examine the

[5] The sample space could serve, reflexively, as a sampling space for the same experiment, because there were equal numbers of green and blue marbles in the box. Also, the ratio of the sample size (4) and the number of marbles in the tub (hundreds) makes negligible the fact that, strictly speaking, the scoop constitutes four single samples of type ‘no-returns.’

corresponding properties of the number line, the mathematical artifact which, as we shall see later, emerged from operating the 4-blocks.

Number Line Requisites: Deconstruction Into Semiotic Fragments Toward Analysis of Situated Emergence in Instrumented Activity

We are exploring the conjecture that the number line, normatively available *a priori* and utilized *for* an activity involving the manipulation of quantitative information, can emerge *from* such situated activity (and henceforth be utilized for this activity). Let us adopt a bricolage view toward the composition of mathematical artifacts so as to investigate how the various elements of an artifact may cohere fortuitously into a system of practice serving goal-oriented problem-solving engagement. That is, we will consider the possibility that some structural and functional aspects of the number line were already embedded as affordances of the available media used in the study—fragmented potentialities that cohered into a clockwork systematic procedure that was realized within the context governed by the study's interview protocol and ultimately served the pedagogical objectives of this design-based research intervention.[6]

To investigative the situated origins of the number line in our interview data, we ought to know what to search out for in our data. Thus we begin by stepping back to determine independent structural and functional elements that characterize the number-line gestalt as we customarily know and use it. Equipped with the list of elements, we will subsequently scout the data for indications of those elements as they emerge in students' interactions.

At the very least, a number-line system should:

- designate upon a medium—paper, screen, or other—a set of inscribed locations, each uniquely associated with one of the lower integers and, by potential extension, given the needs of the particular context, to the higher integers;
- embody rules of designation, such that the mapping between integers and the inscribed locations is non arbitrary; e.g., given the locations of, "1," "2," and "3," we should be able to determine the locations of subsequent integers;

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[6] Note that such "archeological" scrutiny of the emergence of a mathematical artifact explicitly does not assume a teleological perspective by which the students ostensibly know that the sample space will constitute an indicator of the expected outcome distribution—we will follow a student's phenomenology without ascribing to the student understandings that we harbor yet s/he does not yet possess. Thus, when we use sophisticated vocabulary to refer to elements of the student's behavior, we will do so with the explicit understanding that these terms are used in order that we can evaluate for alignment between our pedagogical objectives and the student's learning trajectories; where necessary, we will be careful to underscore the etic/emic tension rising from observations made by the researcher-as-interviewer-as-data-analyst. For example, when we say that a student is engaging in combinatorial analysis of a random generator, we do *not* mean to imply neither that the student is familiar with this nomenclature nor that the student initially recognizes the 'towardness' or 'equipmentality' of the activity; asked what she is doing, the student may very well say that she is "coloring in the squares."

- include each of the integers in its order within the "counting poem"—no integer can be repeated or skipped (cf. Gelman, 1993, for the "counting rules");
- ideally, but possibly not necessarily, the number line should suggest determinable locations for non-integer numbers;
- encode the relative magnitudes of each of the integers as respective distances from a point designated as the origin;
- arrange the locations corresponding to the integers along a linear continuity (possibly from left to right on the horizontal plane); thus, and given the previous attributes, the distances between consecutive integer locations become equal, and the number line is therefore a ratio scale;
- constitute an encoding system for quantitative properties of data under inquiry—once the number line in constructed, it can function as a recipient platform upon which numerical aspects of objects can be plotted by placing the objects (or signs standing in for the objects) at the respective locations.
- function as an *x*-axis within a Cartesian plane. When objects are placed along this number-line-as-axis, they might be stacked one upon the other or, to facilitate quantification (e.g., enumeration), one above the other, stretching vertically and parallel to a present–absent *y*-axis;
- enable the perspicacious eye rapid, vivid extraction of information by discerning quantitative relations within the data, such as central tendencies or trends. A crude Cartesian ploy would be to sort objects by type along the *x*-axis then stack them so that the *y*-axis encodes the number of objects per type.

The above analysis of the elements composing the number line was somewhat decontextualized—it treated the number line as a mathematical artifact that could be instrumentalized within a range of contexts and toward a range of objectives, such as in situated problem-solving. Yet such an "abstract" or unsituated treatment of the number line, though concise and potentially generative, does not lend itself well to addressing issues of conceptual learning in context or issues of educational-design frameworks. Namely, a decontextualized analysis of a mathematical artifact into fragmented affordances does not ask how the artifact becomes consciously or tacitly enlisted by a problem solver toward the completion of a goal-based activity; it doesn't address the question of how the artifact becomes instrumentalized toward the *in-situ* objectives and how such instrumentalization, in turn, might instrument the individual toward the phenomena under inquiry (see Vérillon & Rabardel, 1995, for the triadic Instrumented Activity Situations Vygotskiian model). And yet, I contend, the process by which the number line emerged for our participants as a functional aspect of the available media demonstrates that this emergence was deeply informed *by* and, reflexively, informing *of* students' eventual understanding of the sample space as encoding information relevant to the problem as stated. That is, the assembly process involved more than creating orderly representations—*the construction process became immixed with the design's core content*. In particular, the constitution of the five "things we can get" (the five 4-block outcomes-taken-as-events) as the "tick marks" of the emergent number line both enabled students to build on their intuitive sense that there are five "things we can get" and, in so doing, to begin to explore those "things" as categories rather than just "things" onto themselves. Thus, the dual role of the five "things"—as to-be-sorted *objects* that become also semiotic *marks* along the number-line representation—served both as an enabling feature of the design and, as we shall see, an ambiguity causative of a challenging yet ultimately productive learning issue that was negotiated and resolved by the dyad. This object–mark ambiguity, I wager, may be typical of other learning contexts involving the invention and use of mathematical representations.[7]

In the following section, thus, we will scrutinize the emergence of the number line in a specified context—the construction of a random generator's sample space.

The Number Line Emerges in a Sorting and Assembly Activity

Students participating in our study were asked to use the green and blue crayons and empty 4-block cards to create images of all the possible marble scoops. Most students spontaneously organized the activity by working in sequence on each of the event, no-green, 1-green, 2-green, 3-green, and 4-green, and some students expanded each of these into its permutations before moving on to the next event. By token of thus using the cards' number-of-green value as the organizing property for generating the sample space, students perfunctorily arranged the cards or card sets upon their desks along a rough linear projection, as they moved from group to group. That is, as they completed the construction of each card or set, students wished to demarcate this construction as singular, so they moved their hands away from the constructed group and on to the nearest available physical point, where they conducted the next construction. From this repeated set of actions, a "necklace" of construction islands was born on the desk surface that both traced the order of construction—often "0, 1, 2, 3, 4"—and, presumably, helped students monitor their work (just as people charged with counting a large number of physical items may group them by tens).

Students' emergent subgoal then became to arrange or sort this collection of objects, some of which were already grouped in sets, on the basis of their respective values for the property 'number of green squares.' Apparently some representational/semiotic forms are more suited to reveal, document, and communicate quantitative properties and are therefore socio–cognitively privileged tools for those specific goal-oriented contexts of practice. Specifically, students spontaneously placed the objects-as-categories along a tighter linear concatenation. Thus, a closer-knit construction, still ordered in accord with these objects' values for the target property 'number of green,' emerged.

How did this construction process become intertwined with students' understanding of the content embedded in the design? Below, we examine this question, attending in particular to the confusions experienced in the dyad over the meaning of these co-constructed semiotic objects and how these confusions were resolved in the form students' new understandings of the target content of this design.

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[7] Another example of a sign playing dual roles as a category and as an item within that category is the case of multiplication tables that do not included a set of 1-through-10 factors outside of the 10-by-10 square. In that artifact, the numerals in the top row and in the left-hand column are each both a factor and a product of their multiplication by 1.

Ontological Imperialism as an Enabling Constraint

As suggested earlier, many of the students first created only five cards, because they saw those cards as exhausting the sample space. In Abrahamson, Bryant, Howison, and Relaford–Doyle (2008) we examine why and how the students could possibly see five specific outcomes as five events. For example, students took the outcome with exactly one blue square in the top-right corner to mean *any* outcome with exactly one blue square. Namely, the students were seeing as a combination that which for us was patently a particular permutation—the interviewer and student gazed at the same referent but ascribed different meanings to it. The semiotic approach espoused in this study shifts the analytic perspective away from what an object 'is,' e.g., "one of 16 possible outcomes," to what a particular person intends to express with the object, e.g., "one of the 5 things we can get." Though the researchers may see a 4-block card as a particular outcome, we argue, still for the student the card may constitute an event, because the student intends for it to express an event. Moreover, and poignantly, we propose, the *semiotic means of objectification* (Radford, 2003) made available to the students—namely the empty 2-by-2 cards—tacitly introduced into the learning environment what Bamberger and diSessa (2003) have called *ontological imperialism*: given the 4-block format, even if students so wished to represent an order-less combination of '3 green, 1 blue,' they inadvertently created an ordered outcome—no alternative media was available, such that the students were ensnared into expressing aspects of the random generator through the researchers' ontological lenses. Yet—and herein lay the communication breakdown between the interviewer and student—students persisted in seeing these outcomes as events.

Thus, even once they completed the construction of the sample space and its assembly as the combinations tower, many students would persevere in regarding the five initial cards as "the things we can get" and the other eleven as redundant variants on these five outcomes. Even students who had spontaneously offered the insight that the *entire* combinations tower indexed the relative frequencies they had experienced in the marbles box…even these students still would confuse events and outcomes. Yet though such a view toward the combinations tower was transitional (in the Piagetian sense) and not yet mathematically normative, within the trajectory of the interview-based conceptual microgenesis, this view did confer the five cards as constituting categories and the rest of the cards as members of those categories, a key hierarchical relation between an event and its constitutive outcomes. Thus, a conceptual milestone—the object–mark dual property of each of the five cards—was achieved through the acts of concatenating these cards into a linear sequence in rising order of their respective values for the 'number-of-green-squares' value, then stacking above them the remaining eleven cards.[8]

Summary

Thus, a number line evolved as a consequence of concatenating a series of identical square objects that bore colored spatial configurations each with as many as 0, 1, 2, 3, or 4 green squares. By ordering these objects on the basis of their respective number of

[8] One 6th-grade student did comment that each card in the bottom row was interchangeable with any of the cards above it, yet he too, at that point, thought that only five cards were necessary and that the other eleven were redundant for achieving the goal of warranting the intuitively determined outcome distribution.

green squares, a structure evolved that could subsequently encode the number of green squares in other objects by virtue of laying those objects above their corresponding "tick mark" cards. This process, a case of distributed cognition in action, continued until it exhausted all the cards in the sample space, ultimately creating a symmetrical structure that students identified as resonating with their prior intuitive inference. That is, once the sample space was assembled into the form of the combinations tower, it could take on a new semiotic role—indexing the relative frequencies of the five possible events that students' had evaluated in the context of the marbles-box random generator (see Hutchins, 2005, on material anchors for conceptual blends).

Conclusion

When we pause to think about the number line, we think of it as an instrument supporting mathematical practice; for example, the number line is enlisted as a tool for completing mathematical tasks involving a set of data. To complete such tasks, we regularly inscribe the number line in traditional or computational media; alternatively, the number line is available for tacit mental imagery supporting our mathematical reasoning. Broadly speaking, the number line enhances mathematical practice by spatially encoding cardinal properties of objects as distance from a common origin, very often using a common spatial unit. Once the data is encoded into the number-line system, we utilize cognitive faculties such as perceptually-based proportional reasoning to enable quantitative orientation and visual comparisons; the number line thus facilitates the emergence of perceptually apprehensible quantitative patterns embedded in the data.

In this paper, however, I analyzed empirical examples suggesting the possibility of a reverse order of actions—first the (physical) objects are ordered and only then the number line emerges from quantitative patterns embedded in these objects. Specifically, I discussed findings from microgenetic analyses of problem solving, in which the number line emerged as a functional potentiality of a set of manipulable objects that were arranged on a desk. This microgenesis of a few dozen Bay Area kids' one-hour tinkering, in turn, offers a possible insight into phylogensis over millennia of human endeavor, thus reversing Haeckel's hackneyed dictum. That is, we might consider an 'object before inscription' conjecture pertaining to the historical origin of the number line (see Schmandt-Besserat, 1992, on a similar yet well substantiated conjecture regarding the origin of numerals in the ancient Mesopotamian clay-token accounting system). Namely, the number line may have emerged within ancient cultural practices as an inscribed simulacrum of a sequence of concatenated physical objects such as equidistant tokens arranged along a linear trajectory for the sake of organizing information. Once inscribed on paper-like media, say as stamped icons, the initially ordinal composition of a set of discrete objects could take on the encoding of non-integer quantities in between the icons, and an actual line could connect the icons. In fact, the presence of unaccounted-for spaces in between the integers may give rise to a need for mathematical closure—people might wish to account for the inter-integer locations, just as irrational numbers and complex numbers were discovered because arithmetical operations yielded *numerus incognitus*. And just as the study of etymology demonstrates how phonetically akin verbal units are attracted to, and away from, each other, in the course of the years this new artifact would possibly be attracted to, and differentiated from, instruments of measurement (yardsticks, rulers). One consequence of such a process would be a foregrounding of the distance of marks from the origin rather than from adjacent marks.

An implication of such a conjecture, if supported, and in any case the implications of such a line of reasoning, is that within constructivist learning environments, too, the number line may emerge as a problem-solving device. That is, we could possibly set up activities in which students discover and reflect on the number line as a useful instrument supporting construction and reasoning, for example when students are engaged in explorative data analysis. Such a design would possibly be more effective than current traditional curricula in making properties of the number line evident and meaningful to students, because the students themselves would reinvent and appropriate the number line by discovering and recognizing its affordances.

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